

# Severi varieties

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January 2001

## Abstract

R. Hartshorne conjectured and F. Zak proved (cf [5, p.9]) that any smooth non-degenerate complex algebraic variety  $X^n \subset \mathbf{P}^m$  with  $m < \frac{3}{2}n+2$  satisfies  $\text{Sec}(X) = \mathbf{P}^m$  ( $\text{Sec}(X)$  denotes the secant variety of  $X$ ; when  $X$  is smooth it is simply the union of all the secant and tangent lines to  $X$ ). In this article, I deal with the limiting case of this theorem, namely the Severi varieties, defined by the conditions  $m = \frac{3}{2}n+2$  and  $\text{Sec}(X) \neq \mathbf{P}^m$ . I want to give a different proof of a theorem of F. Zak classifying all Severi varieties. F. Zak proves that there exists only four Severi varieties and then realises a posteriori that all of them are homogeneous; here I will work in another direction : I prove a priori that any Severi variety is homogeneous and then deduce more quickly their classification, satisfying R. Lazarsfeld et A. Van de Ven's wish [5, p.18]. By the way, I give a very brief proof of the fact that the derivatives of the equation of  $\text{Sec}(X)$ , which is a cubic hypersurface, determine a birational morphism of  $\mathbf{P}^m$ .

I wish to thank Laurent Manivel for helping me in writing this article.

## 1 Preliminary facts about Severi varieties and an example

In this section, I state results of F. Zak (cf [5, p.19]). The reader can find the details of their proofs in my preprint written in french (cf [7]).

Let  $X$  be a Severi variety and  $\text{Sec}(X)$  its secant variety. Let, for any  $P \in \text{Sec}(X) - X$ ,  $Q_P := \{x \in X : (xP) \text{ secant or tangent to } X\}$  and  $\Sigma_P := \bigcup_{x \in Q_P} (xP)$ , the union of all secant and tangent lines through  $P$ . It may also be described as the cone on  $P$  with basis  $Q_P$ .

**Theorem 1.1** *For any  $P \in \text{Sec}(X) - X$ ,*

- *$a : Q_P$  is a smooth  $\frac{n}{2}$ -dimensional quadric, and  $\Sigma_P$  is a  $(\frac{n}{2}+1)$ -dimensional linear space.*
- *$b : \Sigma_P \cap X = Q_P$*

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*AMS mathematical classification :* 14M07,14M17,14E07.

*Key-words :* Severi variety, secant variety, Zak's theorem, projective geometry.

- $c : \Sigma_P - X = \{P' \in \text{Sec}(X) - X : T_{P'}\text{Sec}(X) = T_P\text{Sec}(X)\}$
- $d : \forall P' \in \text{Sec}(X) - X, Q_P = Q_{P'} \Leftrightarrow P' \in \Sigma_P$

**Theorem 1.2**  *$\text{Sec}(X)$  is a cubic hypersurface singular exactly on  $X$ .*

Their proofs imply the

**Lemma 1.1** *Let  $P \in \text{Sec}(X) - X$  and  $M$  a  $(\frac{n}{2} + 2)$ -linear space containing  $\Sigma_P$  and not contained in  $T_P\text{Sec}(X)$ . Then there exists  $x \in X - \Sigma_P$  such that*

- (i)  $M \cap X = Q_P \cup \{x\}$
- (ii)  $M \cap \text{Sec}(X) = \Sigma_P \cup C(x, Q_P)$  ( $C(x, Q_P)$  is the cone on  $x$  with base  $Q_P$ )

Now let's study the example of  $\mathbf{P}^2 \times \mathbf{P}^2$ , which is the variety of  $3 \times 3$ -matrices of rank 1 in  $\mathbf{P}^8 = \mathbf{P}[\mathcal{M}_3(\mathbb{C})]$ . It is indeed a Severi variety since matrices in  $\text{Sec}(X)$  have rank at most 2; and we can see from this that  $\text{Sec}(X)$  is defined by the determinant, an equation of degree 3 (cf theorem 1.2). If  $P = M + N$ , with  $M, N \in X$  and  $P \in \text{Sec}(X) - X$ , we have  $\text{rk}(P) = 2$ ,  $\text{Ker}(M) \cap \text{Ker}(N) = \text{Ker}(P)$  and  $\text{Im}(M) + \text{Im}(N) = \text{Im}(P)$ . Let's consider the case where  $P = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}$ , we can then easily compute that :

$$\Sigma_P = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

and

$$Q_P = \Sigma_P \cap \{ad - bc = 0\}$$

This illustrates theorem 1.1. We now go on analysing this particular example so as to prepare the general method : let  $F$  be the symmetric trilinear form such that  $F(M, M, M) = \det(M)$ , namely for  $M_i = (C_{i,j})_{1 \leq j \leq 3}$  any matrix considered as 3 column-vectors ( $i = 1, 2, 3$ ),

$$F(M_1, M_2, M_3) = \frac{1}{6} \sum_{\sigma \in S_3} \det(C_{\sigma(i), i})_i$$

Identifying a matrix with its orthogonal hyperplane for the canonical bilinear form, one can easily compute that  $\tilde{G} : M \rightarrow F(M, M, \cdot)$ ,  $M \in \mathcal{M}_3(\mathbb{C})$  is just the comatrix application and then the fonction  $G$  defined by  $G(M) = \frac{F(M, M, \cdot)}{F(M, M, M)}$  is an involution out of  $\text{Sec}(X)$  as it is the inverse application. On the other side, we may define a regular function on  $\mathbf{P}^m - X$  which coincide with  $G$  on  $\mathbf{P}^m - \text{Sec}(X)$  (by setting  $G(M) = \text{Com}(M)$ ) and this morphism verifies  $G[\text{Sec}(X)] = X$ . We are going to see that all of this can be proved in the general case, except for the fact that we won't identify a priori  $\mathbf{P}^m$  and  $\mathbf{P}^{m*}$ .

We are also going to show that any Severi variety is homogeneous. For any variety  $Z$ , let  $Z^*$  be its dual variety, that is the variety of all hyperplanes tangent to  $Z$  at one point. In the example we have considered, any invertible matrix yields a linear isomorphism

$L_M$  between  $\mathbf{P}^8$  and  $\mathbf{P}^{8*}$  (namely  $L_M(B)$  is the hyperplane in  $\mathbb{C}^9$  orthogonal to the matrix  $-M^{-1}BM^{-1}$  for the canonical scalar product <sup>2</sup>). This isomorphism maps  $X$  on  $\text{Sec}(X)^*$  and, for  $M$  and  $N$  varying, the endomorphisms of  $\mathbf{P}^8$   $(L_N)^{-1} \circ L_M$  restrict to a family of endomorphisms acting transitively on  $X$ , proving  $X$ 's homogeneity. The same will be shown to be true in the general case.

## 2 Homogeneity of Severi varieties

Let  $X^n \subset \mathbf{P}^m$  be any Severi variety,  $Y = \text{Sec}(X)^* \subset \mathbf{P}^{m*}$ , and as before  $F$  a symmetric trilinear form such that  $v \in \text{Sec}(X) \Leftrightarrow F(v, v, v) = 0$ . We will simplify the expression  $F(v, v, v)$  in  $F(v)$ . From now on, all elements will be considered as elements of  $\mathbb{C}^{m+1}$  (and not  $\mathbf{P}^m$ ) and we will denote the same way varieties in  $\mathbf{P}^m$  and their cones in  $\mathbb{C}^{m+1}$ . For  $w_0 \notin \text{Sec}(X)$  and  $w \in \mathbb{C}^{m+1}$  let us denote by  $L_{w_0}(w)$  the linear form  $2F(w_0)F(w_0, w, \cdot) - 3F(w_0, w_0, w)w_0^*$ , where  $w_0^*$  stands for the linear form  $F(w_0, w_0, \cdot)$ . Let us also notice that  $L_{w_0}(w)$  and the differential  $D_{w_0}G(w)$  are colinear if  $G$  stands for the rational map  $w \mapsto \frac{w^*}{F(w)}$  we have just mentionned.

Let  $w_0$  be fixed and  $x \in X$  such that  $w_0^*(x) \neq 0$ . I am going to explain the geometric meaning of  $L_{w_0}(x)$ . Taking into account the fact that  $x^* = 0$  ( $X$  is the singular locus of  $\text{Sec}(X)$ ), we know that  $x + \lambda w_0 \in \text{Sec}(X)$  if and only if  $\lambda = 0$  or  $\lambda = \lambda_x := -\frac{w_0^*(x)}{F(w_0)}$ . On the other direction, let  $H_{w_0}(x)$  be the tangent space to  $\text{Sec}(X)$  at  $x + \lambda_x w_0$ . The equation of this hyperplane is  $F(x + \lambda_x w_0, x + \lambda_x w_0, \cdot) = 0$ , or  $L_{w_0}(x)(\cdot) = 0$ .

So  $D_{w_0}G(x)$  is the equation of the tangent space to  $\text{Sec}(X)$  at the other intersection point of  $(xw_0)$  with  $\text{Sec}(X)$  if  $x$  stands outside the hyperplane  $w_0^* = 0$ . So  $D_{w_0}G(X) \subset Y$ . In the opposite direction, if  $P \in \text{Sec}(X) - X$  verifies  $w_0 \notin H_P := T_P \text{Sec}(X)$ , we know from lemma 1.1 that there exists a point  $x_P$  in  $X \cap (\Sigma_P + w_0) - Q_P$ , and then  $x_P$  is such that  $D_{w_0}G(x_P) = H_P$ . As  $Y$  is known to be non degenerate,  $D_{w_0}G(X)$  contains an open subset of  $Y$  and we have the

**Proposition 2.1** *If  $w_0 \notin \text{Sec}(X)$ ,  $D_{w_0}G$  is a linear isomorphism from  $\mathbf{P}^m$  to  $\mathbf{P}^{m*}$  such that  $D_{w_0}G(X) = Y$ .*

**Corollary 2.2**  *$X$  is homogeneous.*

**Proof:** Let  $x, x' \in X$ ; it is enough to find  $w, w' \notin \text{Sec}(X)$  such that  $D_w G(x) = D_{w'} G(x')$ . Let  $H_P$  containing neither  $x$  nor  $x'$ ;  $w$  in  $\Sigma_P + x$  and  $w'$  in  $\Sigma_P + x'$  achieve it. •

## 3 Classification of Severi varieties

We now want to prove Zak's classification theorem (cf [5])

**Theorem 3.1** *There are only four Severi varieties, namely :*

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<sup>2</sup>  $L_M(B) = dG_M(B)$ ; in the second part, we are going to see the interest of this fonction.

- $n=2 : X = \mathcal{V} \subset \mathbf{P}^5$ , the Veronese surface
- $n=4 : X = \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$
- $n=8 : X = G(2,6) \subset \mathbf{P}^{14}$
- $n=16 : X = \mathcal{E}_6 \subset \mathbf{P}^{26}$

The reader may find in [5, p.13] elements to understand the construction of  $\mathcal{E}_6$  and the fact that it is homogeneous under  $E_6$ .

**Proof :** Let  $V = \mathbb{C}^{m+1}$ . Let  $\mathcal{H}$  be the group of automorphisms of  $\mathbf{P}V$  preserving  $X$ . As  $\mathcal{H}$  acts transitively on the non-degenerate homogeneous projective variety  $X$ , it is a semi-simple subgroup of  $PGL(V)$ . Let  $\pi$  be the projection  $SL(V) \rightarrow PSL(V)$  and  $\mathcal{G}$  the identity component of  $\pi^{-1}(\mathcal{H})$ .  $\mathcal{G}$  is again a semi-simple subgroup, and  $V$  is an irreducible  $\mathcal{G}$ -module.

If  $\mathcal{G}$  is not simple, we may write  $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$  with non-trivial  $\mathcal{G}_1$  and  $\mathcal{G}_2$  acting on  $V_1$  and  $V_2$  such that  $V = V_1 \otimes V_2$ . Let  $n_i := \dim X_i$  and  $n_i + \delta_i := \dim V_i$  (then  $\delta_i \geq 1$ ). For  $X = X_1 \times X_2$  to be a Severi variety, we need  $\frac{3}{2}(n_1 + n_2) + 3 = (n_1 + \delta_1) \times (n_2 + \delta_2)$ , which leads to :  $(n_1 n_2) + (\delta_2 - \frac{3}{2})n_1 + (\delta_1 - \frac{3}{2})n_2 + (\delta_1 \delta_2 - 3) = 0$ .

If  $\delta_1 \geq 2$  and  $\delta_2 \geq 2$ , all the terms of this sum are positive; we can then suppose  $\delta_2 = 1$ , which gives  $(\delta_1 - \frac{3}{2})n_2 + (n_2 - \frac{1}{2})n_1 + \delta_1 - 3 = 0$  and so  $\delta_1 \leq 2$ .

If  $\delta_1 = 2$ , the equation gives  $n_1(n_2 - \frac{1}{2}) + \frac{n_2}{2} - 1 = 0$  so  $n_1 = n_2 = 1$ . In this case we get  $X = \mathbf{P}^1 \times \nu_2(\mathbf{P}^1) \subset \mathbf{P}^1 \times \mathbf{P}^2 \subset \mathbf{P}^5$ , which is not a Severi variety.

If  $\delta_1 = 1$ , we get  $(2n_1 - 1)(2n_2 - 1) = 9$  and, if  $n_1 \leq n_2$ ,  $n_1 = n_2 = 2$  (we then get  $\mathbf{P}^2 \times \mathbf{P}^2$ ) or  $n_1 = 1$  and  $n_2 = 5$ ; in this case the variety would be  $\mathbf{P}^1 \times \mathbf{P}^5$  which is not a Severi variety.

If  $\mathcal{G}$  is simple, we take a Borel subgroup  $\mathcal{B} \subset \mathcal{G}$  and a maximal torus  $\mathcal{T} \subset \mathcal{B}$ ; let  $\mathfrak{g}$  be the Lie algebra associated to  $\mathcal{G}$  and  $\mathfrak{h}$  the Cartan subalgebra associated to  $\mathcal{T}$ . Let then  $\Delta$  be the root system, and  $\Delta^+$  the set of all positive roots, which by definition we ask to correspond to  $\mathcal{B}$ . Let  $\lambda$  be the highest weight of  $V$ ,  $\mu = w_0(\lambda)$  the lowest weight,  $l_\lambda$  and  $l_\mu$  the associated lines of eigenvectors. Let  $e_\lambda \in l_\lambda$  and  $e_\mu \in l_\mu$  different from 0 and  $s = e_\lambda + e_\mu$ .

**Lemma 3.1** *Sec(X) has finitely many  $\mathcal{G}$ -orbits, and  $s$  is in the open one.*

**Proof :** By the Bruhat decomposition theorem, we see that  $X \subset \mathbf{P}^m$  has only finitely many  $\pi(\mathcal{B})$ -orbits so  $Sec(X)$  has only finitely many  $\mathcal{G}$ -orbits. Let  $\mathcal{B}_\lambda$  and  $\mathcal{B}_\mu$  be the stabilizers of  $l_\lambda$  and  $l_\mu$ . Let for any root  $\alpha$ ,  $\mathfrak{g}_\alpha$  be the associated root space of  $\mathfrak{g}$ . Letting  $\mathfrak{n}^+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$  and  $\mathfrak{n}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$ , we know that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$  and that the Lie algebras associated to  $\mathcal{B}_\lambda$  and  $\mathcal{B}_\mu$  are  $\mathfrak{h} \oplus \mathfrak{n}^+$  and  $\mathfrak{h} \oplus \mathfrak{n}^-$ , so that  $\mathcal{B}_\lambda \mathcal{B}_\mu$  is a dense subset of  $\mathcal{G}$ . Thus  $\mathcal{B}_\mu \cdot l_\lambda = \mathcal{B}_\mu \mathcal{B}_\lambda \cdot l_\lambda$  contains an open subset  $U$  of  $X = \mathcal{G} \cdot l_\lambda$ . Consequently, the  $\mathcal{G}$ -orbit of  $l_\lambda \times l_\mu$  in  $X \times X$  contains  $U \times l_\mu$  and so  $\mathcal{G} \cdot (U \times l_\mu)$ , which is dense in  $X \times X$ .

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Terracini's lemma (cf for example [6]) allows us to conclude that  $T_s Sec(X) = \langle T_{e_\lambda} X, T_{e_\mu} X \rangle$ .

As  $X = \mathcal{G}.l_\lambda$ , denoting by  $V_\alpha \subset V$  the weight subspace associated to the weight  $\alpha$ ,

$$T_{e_\lambda}X = Ad(\mathfrak{g})l_\lambda = l_\lambda \oplus (\oplus_{\alpha \in \Delta^+} Ad(\mathfrak{g}_{-\alpha})l_\lambda) \subset l_\lambda \oplus (\oplus_{\alpha \in \Delta^+} V_{\lambda-\alpha})$$

Similarly,

$$T_{e_\mu}X \subset l_\mu \oplus (\oplus_{\alpha \in \Delta^+} V_{\mu+\alpha})$$

As  $\dim Sec(X) < 2 \dim X + 1$ , there exists  $\alpha, \beta \in \Delta^+$  such that  $\lambda - \alpha = \mu + \beta$ , that is  $\lambda - w_0(\lambda) = \alpha + \beta$ . Let's notice that if  $\lambda$  is the highest root, this equation has only one solution :  $\alpha = \lambda$  and  $\beta = -w_0(\lambda) = \lambda$ . The variety we get is then the closed orbit of the action of  $\mathcal{G}$  in its projectiviced adjoint representation ; it is not a Severi variety as  $T_{e_\lambda}X \cap T_{e_\mu}X$  equals the line  $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]$  of  $V_0$  and so  $\dim Sec(X) = 2 \dim X$ .

In the general case, the existence of two positive roots whose sum is  $\lambda - w_0(\lambda)$  is a very restrictive condition which allows us to easily classify all homogeneous varieties whose secant variety does not have maximal dimension.

I am now going to study two examples, a classical and an exceptional one, of resolution of the equation  $\lambda - w_0(\lambda) = \alpha + \beta$ . I will use N. Bourbaki's notations and results (cf [1]).

– **root system of type  $A_n$**

Here roots live in  $\mathbb{C}^{n+1}$  and the positive roots are the  $\epsilon_i - \epsilon_j, 1 \leq i < j \leq n+1$ . The fundamental weights are

$$\omega_i = (\epsilon_1 + \dots + \epsilon_i) - \frac{i}{n+1} \sum_{j=1}^{n+1} \epsilon_j$$

and  $w_0 = -Id$ . Thus  $\omega_i - w_0(\omega_i) = \epsilon_1 + \dots + \epsilon_i - \epsilon_{n+1-i} - \dots - \epsilon_{n+1}$  if  $i < \frac{n+1}{2}$  and  $\omega_{n+1-i} - w_0(\omega_{n+1-i}) = \omega_i - w_0(\omega_i)$ . On the other hand, if  $\alpha$  and  $\beta$  are positive roots, and if  $\alpha + \beta = \sum s_i \epsilon_i$ , then  $\sum |s_i| \leq 4$ . As  $\lambda$  is a sum of fundamental weights, we can deduce that only four cases can occur :

- $\lambda = \omega_1$  or  $\lambda = \omega_n$ . Then  $X = \mathbf{P}^n$ .
- $\lambda = 2\omega_1$  or  $\lambda = 2\omega_n$ . Then  $X = \mathbf{P}^n \subset \mathbf{P}S^2\mathbb{C}^{n+1}$ .
- $\lambda = \omega_2$  or  $\lambda = \omega_{n-1}$ . Then  $X = G(2, n+1) \subset \mathbf{P}\Lambda^2\mathbb{C}^{n+1}$ .
- $\lambda = \omega_1 + \omega_n$  : adjoint representation.

– **root system of type  $E_6$**

In this case, roots live in  $\mathbb{C}^8$  and in the canonical basis  $(\epsilon_i)$ , the positive roots are the  $\pm\epsilon_i + \epsilon_j$  ( $1 \leq i < j \leq 5$ ) and the  $\frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^5 (-1)^{\nu_i} \epsilon_i)$  with  $\sum_{i=1}^5 \nu_i$  even.

Let's consider the application which sends  $i$  to the sequence  $(\lambda_j^i)_j$  such that if  $(\omega_i)$  are the fundamental weights,  $\omega_i + w_0(\omega_i) = \sum_j \lambda_j^i \epsilon_j$ . This application is given by :

$$\begin{aligned}
1 &\mapsto (0, 0, 0, 0, 1, -1, -1, 1) \\
2 &\mapsto (1, 1, 1, 1, 1, -1, -1, 1) \\
3 &\mapsto \frac{1}{2}(-1, 1, 1, 3, 3, -3, -3, 3) \\
4 &\mapsto (0, 0, 2, 2, 2, -2, -2, 2) \\
5 &\mapsto \frac{1}{2}(-1, 1, 1, 3, 3, -3, -3, 3) \\
6 &\mapsto (0, 0, 0, 0, 1, -1, -1, 1)
\end{aligned}$$

Given the formula for the positive roots,  $(\alpha + \beta)_8$  can only be  $0, \frac{1}{2}$  or  $1$  and as  $\forall i, (\omega_i + w_0(\omega_i))_8 > 0$ , one can deduce that if  $\lambda = \sum \lambda_i \omega_i$ , only one  $\lambda_i$  may be different from 0 and then equals 1. Finally, one has the list :

- $\lambda = \omega_1$  or  $\lambda = \omega_6$  : variety  $\mathcal{E}_6$ .
- $\lambda = \omega_2$  : adjoint representation.

One can treat all the cases this way, and check that the homogeneous varieties whose secant variety does not have the maximal dimension are either adjoint varieties,  $X = \mathbf{P}^n$ , quadrics, Veronese mappings of  $\mathbf{P}^n$  into  $\mathbf{P}S^2\mathbb{C}^{n+1}$ , grassmannians of 2-planes, possibly annihilating a quadratic or symplectic form, or the closed orbit of the minimal representation of  $E_6, F_4$  (a hyperplane section of that of  $E_6$ ) or  $G_2$  ( $X$  is then also a quadric). An immediate computation of dimension then concludes the proof.

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## 4 More geometric properties

We are now going to show that  $G$  exchanges  $X$  and  $X^*$ , and is a birational involution. I want to prove these properties without using the previous classification.

As  $D_{w_0}G$  is a linear isomorphism between  $X$  and  $Y$ ,  $Y$  is a Severi variety . So let  $F^*$  be a trilinear symmetric form such that  $l \in \text{Sec}(Y) \Leftrightarrow F^*(l, l, l) = 0$  and  $l \in Y \Leftrightarrow F^*(l, l, \cdot) = 0$ . Let  $G^*(l) = \frac{F^*(l, l, \cdot)}{F^*(l, l, l)}$ .

**Proposition 4.1** *Let  $w_0 \in \mathbb{C}^{m+1} - \text{Sec}(X)$ . Then  $G^* \circ G(w_0) = w_0$ .*

In particular  $G$  defines a birational map from  $\mathbf{P}^m$  to  $\mathbf{P}^{m*}$  with inverse  $G^*$ . The details of the proof of this are left to the reader in [4, p.79] and written by L. Ein and N. Shepherd-Barron ([2, p.78, Theorem 2.6]). They give two proofs but both of them require to know that  $X$  is homogeneous ; and I pretend to give more elementary arguments.

**Proof :** As  $u \in \text{Sec}(X) \Leftrightarrow L_{w_0}(u) \in \text{Sec}(Y)$ , there exists  $\lambda_{w_0} \in \mathbb{C}^*$  such that  $F^*[L_{w_0}(u), L_{w_0}(v), L_{w_0}(w)] = \lambda_{w_0} F(u, v, w) \ (\diamond)$

Let  $w_0 \in \mathbb{C}^{m+1} - \text{Sec}(X)$  and  $l \in (\mathbb{C}^{m+1})^*$ , we want to see that

$$\frac{l(w_0)}{F(w_0)} = \frac{F^*(w_0^*, w_0^*, l)}{F^*(w_0^*)}$$

As  $L_{w_0}(w_0) = -F(w_0)w_0^*$ , applying  $(\diamond)$  with  $u = v = w_0$  and  $w$  such that  $L_{w_0}(w) = l$  yields :

$F^*(w_0^*, w_0^*, l)F^2(w_0) = \lambda_{w_0}F(w_0, w_0, w)$ . If we take  $u = v = w = w_0$ , we get :  
 $F^*(w_0^*, w_0^*, w_0^*)F^3(w_0) = -\lambda_{w_0}F(w_0)$ . The quotient of the last two equalities yields the expected equality.

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$G$  considered as a rational morphism can be extended to  $\text{Sec}(X) - X$  by  $G(p) = p^*$  and then  $G(\text{Sec}(X) - X) = Y$ . Let's denote by  $G(X)$  the total transform of  $X$ , that is the set of limits of  $G(x_n)$  for  $(x_n)$  any sequence converging to an element of  $X$ . The next proposition is stated without proof in [4, p.79] :

**Proposition 4.2**  $G(X) = \text{Sec}(Y) = X^*$ .

**Proof :** We already know that  $G(X) \subset \text{Sec}(Y)$  since out of  $\text{Sec}(Y)$ ,  $G$  is invertible. In the other direction, if  $y_n \rightarrow y$  with  $y \in \text{Sec}(Y) - Y$  and  $y_n \notin \text{Sec}(Y)$ , then letting  $x_n = G^*(y_n)$ , proposition 4.1 yields  $G(x_n) = y_n$ , so  $G(x_n) \rightarrow y$ . As  $x_n \rightarrow G(y) \in X$ ,  $y \in G(X)$ , and  $G(X) = \text{Sec}(Y)$ .

As far as the second equality is concerned, let  $w \in \mathbf{P}^m - \text{Sec}(X)$  and  $w' = G(w)$ . We have isomorphisms  $D_w G$  and  $D_{w'} G^*$  respectively between  $X$  and  $Y$ , and between  $Y$  and  $(\text{Sec}(Y))^*$ . So we get a composed isomorphism  $X \simeq (\text{Sec}(Y))^*$ . But by the choice of  $w$  and  $w'$  and proposition 4.1, this composed isomorphism is the identity, so that  $X = (\text{Sec}(Y))^*$  and  $X^* = \text{Sec}(Y)$ .

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We now show that  $\text{Sec}(X) - X$  is homogeneous. This implies that under the action of  $\mathcal{G}$ ,  $\mathbb{C}^{m+1}$  has only one invariant,  $F$ , meaning that every  $\mathcal{G}$ -invariant polynomial is, up to a constant, a power of  $F$ . We can then use the classification of all groups whose invariant algebra is free (cf [3]) to deduce another proof of the classification of Severi varieties (theorem 3.1).

Let  $p \in \text{Sec}(X) - X$  and  $P(w_0) = 2F(w_0)p - 6w_0^*(p)w_0$ . We start with two technical lemmas :

**Lemma 4.1** *There exists  $w_0$  such that  $P(w_0) \notin \text{Sec}(X)$ .*

**Proof :** If  $F[P(w)] = 0$  were true for all  $w \in \mathbb{C}^{m+1}$ , then we would get, either  $F(w)^2 F(w, w, p) F(w, p, p) = 0$  for all  $w$  and so  $F(w, w, p) = 0$  for all  $w$ , either  $F(w, p, p) = 0$  for all  $w$ . In the first case, polarisation yields also  $F(w, p, p) = 0$ ; so in both cases  $p \in X$ , contradicting the choice of  $p$ .

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**Lemma 4.2** *Let  $\omega \notin \text{Sec}(X)$ . The matrix  $F(\omega, ., .)$  is invertible.*

**Proof :** This matrix is the differential in  $\omega$  of  $H : w \rightarrow F(w, w, .)$ . As  $F(\omega) \neq 0$ , we can define in a neighbourhood of  $\omega$  a square root  $\sqrt{F(w)}$ , wich is not 0, and proposition 4.1 yields that the composition

$$w \mapsto w_0 := \frac{w}{\sqrt{F(w)}} \xrightarrow{H} F(w_0, w_0, .) \xrightarrow{G^*} G^*[F(w_0, w_0, .)]$$

is the identity, so that  $F(\omega, ., .)$  is invertible.

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**Proposition 4.3** *Sec(X) − X is homogeneous.*

**Proof :** Let  $p \in \text{Sec}(X) - X$  and  $w_0$  such that  $P(w_0) \notin \text{Sec}(X)$ . Let  $L(w)$  the derivative of  $L_\omega(p)$  when  $\omega$  tends to  $w_0$  in the direction of  $w$  :

$$\begin{aligned} L(w) &= 6F(w_0, w_0, w)F(w_0, p, \cdot) + 2F(w_0, w_0, w_0)F(w, p, \cdot) \\ &\quad - 6F(w_0, w, p)F(w_0, w_0, \cdot) - 6F(w_0, w_0, p)F(w_0, w, \cdot) \end{aligned}$$

I'm going to explain why  $\text{Ker}(L) \cap \text{Ker}(w_0^*) = \{0\}$ , so that  $L$  has rank at least  $m$ . Therefore, if  $p$  and  $p'$  are elements of  $\text{Sec}(X) - X$ ,  $\{L_w(p)\}$  and  $\{L_w(p')\}$  contain open subsets of  $\text{Sec}(Y)$  so their intersection is not empty, concluding the proof of the proposition.

So let  $w$  such that  $L(w) = 0$  and  $F(w_0, w_0, w) = 0$ .  $L(w).w_0 = 0$  yields  $F(w_0, w, p) = 0$ ; as  $F(w_0, w_0, w) = 0$ , we deduce  $F[P(w_0), w, \cdot] = 0$ . As by definition of  $w_0$ ,  $P(w_0) \notin \text{Sec}(X)$ , lemma 4.2 yields  $w = 0$ .

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